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Triple point cancelling numbers of surface links and quandle cocycle invariants

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Abstract

The unknotting or triple point cancelling number of a surface link F is the least number of 1-handles for F such that the 2-knot obtained from F by surgery along them is unknotted or pseudo-ribbon, respectively. These numbers have been often studied by knot groups and Alexander invariants. On the other hand, quandle colorings and quandle cocycle invariants of surface links were introduced and applied to other aspects, including non-invertibility and triple point numbers. In this paper, we give lower bounds of the unknotting or triple point cancelling numbers of surface links by using quandle colorings and quandle cocycle invariants.

MSC: 57Q45

Keywords: Surface link; Unknotting number; Triple point cancelling number

1. Introduction

A *surface link* S is a locally flat closed oriented surface in Euclidean 4-space. When S is connected, it is called a *surface knot*. When S is a 2-sphere, it is also called a 2-*knot*. A surface knot S is *unknotted* if S bounds a handlebody in $\mathbf{R}^4$, and a surface link S is *unknotted* if S is the split union of unknotted surface knots. A *diagram* for a surface link will be defined in Section 2. A surface link S is a *pseudo-ribbon* if there is a diagram of S without triple points. It is known that any surface link S can be transformed to an unknotted one, or a pseudo-ribbon, by attaching a finite number of 1-handles to S (cf. [4,12,16]). The *unknotting number* $u(S)$, and the *triple point cancelling number* $\tau(S)$, of S is defined to be the least number of such 1-handles, respectively (cf. [13,16–18,20,22,27]). By definition, the inequality $\tau(S) \leqslant u(S)$ holds.

For surface links S and S', we denote the split union and a connected sum of S and S' by $S \amalg S'$ and $S \sharp S'$, respectively. The split union and a connected sum of n copies of a surface link S are denoted by $\coprod_n S$ and $\sharp_n S$, respectively. It is easy to see that $u(S \sharp S') \leqslant u(S) + u(S')$ and $\tau(S \sharp S') \leqslant \tau(S) + \tau(S')$.

R.H. Fox [11] introduced the notion of p-colorings for classical links which is the same notion of colorings by the dihedral quandle of order p (cf. [6,8,24]). The notion of p-colorings of surface links are also defined similary and will be given in Section 2. When all sheets are colored by the same element, we call the coloring a trivial coloring, which

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is also regarded as a p-coloring in this paper. (It is not in [11].) From now on, we always assume that p is an odd prime integer. We denote the set of p-colorings of a surface link S by $\mathrm{Col}_p(S)$, which is isomorphic to $\mathbf{Z}_p^m$ as linear space for some integer m, where $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$. Its linear space structure is given in Section 2.

Remark 1. Miyazaki [22] proved that $u(\sigma_\alpha \sharp \sigma_{\alpha+2}) = 1$ where σ_α is a spun 2-bridge knot $S(\alpha, 1)$ in Schubert form for odd α. (See [2] for the definition of spun knots.) Using Corollary 19 in Section 2, we have $u(\sigma_\alpha \sharp \sigma_{\alpha'}) = 2$ for $(\alpha, \alpha') \neq 1$ (see Example 23).

These concepts will be explained in Section 2.

For a quandle X, the *associated group*, $G_X = \langle x \in X \mid x * y = yxy^{-1} \rangle$, was introduced in [8,15,21]. The quandle cocycle invariants Φ_κ were defined by using 3-cocycles κ valued in a G_X-module M (cf. [1,5–7]). The values of a quandle cocycle invariant are regarded as multi-sets of elements of M where repetitions of the same elements are allowed. For an element $g \in M$ and a multi-set A, let $a_g(A)$ be the number of g in A. And let $O_\kappa(S)$ be the set of X-colorings which contribute 0 in $\Phi_\kappa(S)$ where 0 is the identity element of M. By definition, $|O_\kappa(S)| = a_0(\Phi_\kappa(S))$. The following two theorems are our main results.

Theorem 2. *Let S be a surface link and let κ be a 3-cocycle of the dihedral quandle X of order p valued in a G_X-module M. And let m be an integer such that the set of p-colorings of S is isomorphic to $\mathbf{Z}_p^m$. If $p^{m-l} > a_0(\Phi_\kappa(S))$ for some $l \in \mathbf{Z}$, then $l + 1 \leqslant \tau(S)$.*

Theorem 3. *Let S and S' be surface links and κ be a 3-cocycle of the dihedral quandle X of order p valued in a G_X-module M. And let m, m' be integers such that the set of p-colorings of S and S' are isomorphic to $\mathbf{Z}_p^m$ and $\mathbf{Z}_p^{m'}$, respectively. If $O_\kappa(S \amalg S')$ forms a linear subspace of $\mathrm{Col}_p(S \amalg S')$ and $p^{m+m'-l} > a_0(\Phi_\kappa(S \amalg S'))$ for some $l \in \mathbf{Z}$, then $l + 1 \leqslant \tau(S \sharp S')$.*

Remark 4. Let S, S', κ, m and m' be as in Theorem 3. Then, $\mathrm{Col}_p(S \amalg S') \cong \mathbf{Z}_p^{m+m'}$. By Theorem 2, if $p^{m+m'-l} > a_0(\Phi_\kappa(S \amalg S'))$ for some $l \in \mathbf{Z}$, then $l + 1 \leqslant \tau(S \amalg S')$, and hence $l \leqslant \tau(S \sharp S')$ (see Lemma 5(2)). Therefore, Theorem 3 gives a better lower bound than this obvious application of Theorem 2.

The following lemma is easily seen.

Lemma 5. *Let S and S' be surface links. Then,*

(1) $\tau(S \sharp S') \leqslant \tau(S \amalg S')$.
(2) $\tau(S \amalg S') \leqslant \tau(S \sharp S') + 1$.

Corollary 6. *Let κ be a 3-cocycle of the dihedral quandle X of order p valued in a G_X-module M. Let S and S' be surface knots with $u(S) = u(S') = 1$ such that $|\mathrm{Col}_p(S)| = |\mathrm{Col}_p(S')| = a_0(\Phi_\kappa(S \amalg S')) = p^2$. Then $\tau(S \sharp S') = u(S \sharp S') = 2$.*

Corollary 7. *Let κ be a 3-cocycle of the dihedral quandle X of order p valued in a G_X-module M. Let S be a surface knot with $u(S) = 1$ such that $|\mathrm{Col}_p(S)| = p^2$ and $a_0(\Phi_\kappa(\coprod_n S)) = p^n$. Then $\tau(\sharp_n S) = u(\sharp_n S) = n$.*

Examples of these corollaries are given in Section 4.

Remark 8. Using Corollary 6, we see that there are infinitely many pairs (S, S') of surface knots such that $\tau(S \sharp S') = \tau(S) + \tau(S')$ (cf. [17,20]). See Section 4.

In Section 2, we will study the set of p-colorings. We will recall the quandle cocycle invariants in Section 3. Our main results are proved in Section 4 and some examples are also given there.

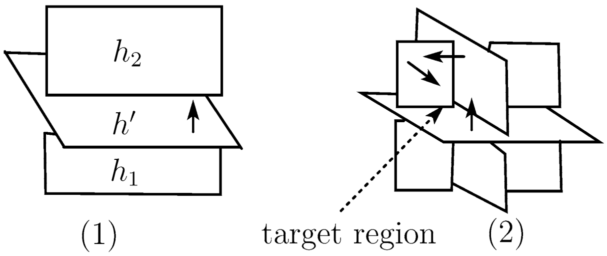

Fig. 1.

2. Quandle colorings of surface links

A *quandle* (cf. [8,15,19,21]) is a set X with a binary operation $*: X \times X \to X$ satisfying the following properties:

(Q1) For any $x \in X$, $x * x = x$.
(Q2) For any $x_1, x_2 \in X$, there is a unique $x_3 \in X$ such that $x_1 = x_3 * x_2$.
(Q3) For any $x_1, x_2, x_3 \in X$, $(x_1 * x_2) * x_3 = (x_1 * x_3) * (x_2 * x_3)$.

Example 9. The set $\mathbf{Z}_p$ is a quandle under the binary operation $a * b = 2b - a$, which is called the *dihedral quandle* of order p and denoted by R_p.

For a surface link S in $\mathbf{R}^4$, modifying it slightly if necessary, we may assume that the projection $\pi: S \to \mathbf{R}^3$ is a generic map. The singularity of the projection consists of double point curves, isolated triple points and isolated branch points. Removing a small regular neighborhood of the under-curve of the double curve, we have a compact surface in $\mathbf{R}^3$. We call it a *diagram* for the surface link S. *Sheets* are connected components of a diagram. Let D be the diagram of S, and $\Sigma(D)$ the set of sheets of D. Using the orientation of S and $\mathbf{R}^3$, we give an orientation normal of each sheet of D. In a neighborhood of each triple point, there are eight regions that are separated by the sheets of D. The region into which normals point is called the *target* region of a given triple point (Fig. 1(2)). Along each double point curve d, the *sheet triple* around d is the triple (h_1, h_2, h') where h_1 and h_2 are the under-sheets and h' is the over-sheet such that the orientation normal of h' points from h_1 to h_2. See Fig. 1(1).

A map $C: \Sigma(D) \to X$ to a quandle X is a *X-coloring* of D if for the sheet triple (h_1, h_2, h') around each d, $C(h_1) * C(h') = C(h_2)$ (Fig. 1(1)). We denote the set of all X-colorings of D by $\mathrm{Col}_X(D)$. For two diagrams D and D' representing the same surface link S, there is a one-to-one correspondence between $\mathrm{Col}_X(D)$ and $\mathrm{Col}_X(D')$ through Roseman moves, which are analogues of Reidemeister moves for surface knots and links. Hence, we also denote it by $\mathrm{Col}_X(S)$. This is equal to the set of quandle homomorphisms from the fundamental quandle of S to X (cf. [8,15]). We remark that $\mathrm{Col}_{R_p}(D) = \mathrm{Col}_p(D)$ (i.e. $\mathrm{Col}_{R_p}(S) = \mathrm{Col}_p(S)$).

Lemma 10. *Let D be a diagram of a surface link S. Then, the set of p-colorings of D forms a linear space* (*over* $\mathbf{Z}_p$) *which is isomorphic to $\mathbf{Z}_p^m$ for some integer m with $k - s \leqslant m \leqslant k$, where k is the number of sheets of D and s is the number of connected components of double curves excluding triple points.*

Proof. We regard $\mathrm{Map}(\Sigma(D), \mathbf{Z}_p)$, the set of all maps from $\Sigma(D)$ to $\mathbf{Z}_p$, as a linear space over $\mathbf{Z}_p$ by $(f + f')(h) = f(h) + f'(h)$ and $(af)(h) = a(f(h))$ in $\mathbf{Z}_p$ where $h \in \Sigma(D)$, $a \in \mathbf{Z}_p$. Let $h_1, \ldots, h_k$ be sheets of D and $d_1, \ldots, d_s$ be the double curves. Then, $\mathrm{Map}(\Sigma(D), \mathbf{Z}_p)$ is isomorphic to the linear space spanned by $\{h_1, \ldots, h_k\}$ over $\mathbf{Z}_p$. We denote it by $\langle h_1, \ldots, h_k \rangle_p$. For each double curve d_i, whose sheet triple is $(h_{i_1}, h_{i_2}, h_{i_3})$, the condition $h_{i_1} * h_{i_3} = h_{i_2}$ in R_p implies a relator $r_i = -h_{i_1} - h_{i_2} + 2h_{i_3}$. Therefore, $\mathrm{Col}_p(D) \cong \langle h_1, \ldots, h_k \mid r_1, \ldots, r_s \rangle_p$ is a linear space isomorphic to $\mathbf{Z}_p^m$ for some integer m with $k - s \leqslant m \leqslant k$. $\square$

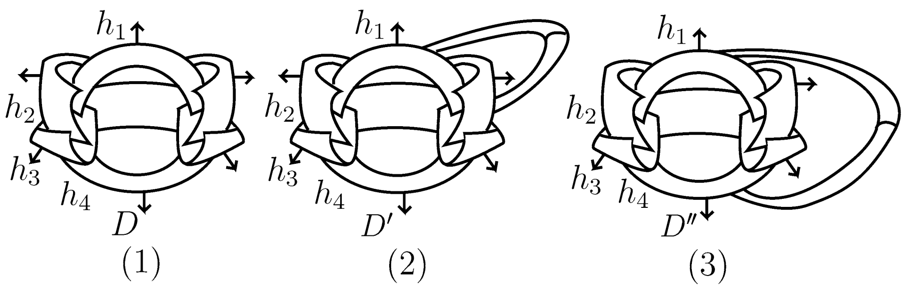

Fig. 2.

Example 11. Let S be a spun trefoil knot and D be a diagram of S illustrated in Fig. 2(1). Then, $\mathrm{Col}_3(D) \cong \langle h_1, h_2, h_3, h_4 \mid r_1, r_2, r_3\rangle_3$ where $r_1 = 2h_1 - h_2 - h_3, r_2 = -h_1 + 2h_2 - h_3$ and $r_3 = -h_2 + 2h_3 - h_4$. Now, let A be a $(3,4)$-matrix over $\mathbf{Z}_3$ given by:

$$A = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \end{pmatrix}.$$

Then, $\dim\langle r_1, r_2, r_3\rangle_3 = \operatorname{rank} A = 2$. Therefore, $\mathrm{Col}_3(D) \cong \mathrm{Col}_3(S) \cong \mathbf{Z}_3^2$.

Lemma 12. *Let S and S' be surface links such that S' is obtained from S by attaching a 1-handle $\widetilde{H}$. Then, there are a diagram D of S, a diagram D' of S' and a 1-handle H in $\mathbf{R}^3$ such that D' is obtained from D by attaching H.*

Proof. Moving the surface link S and 1-handle $\widetilde{H}$ by an ambient isotopy in $\mathbf{R}^4$, we obtain such a diagram D of S, a diagram D' of S' and a 1-handle H. (See [4,12].) □

Let D, D' and H be as in Lemma 12. Let E_1 and E_2 be sheets in $\Sigma(D)$ such that one attaching disk of H is in E_1 and the other is in E_2, and let E' be a sheet in $\Sigma(D')$ such that the belt sphere of H is in E'. Then, $\Sigma(D') = (\Sigma(D)\backslash\{E_1, E_2\}) \cup \{E'\}$. A surjective map $\pi : \Sigma(D) \to \Sigma(D')$ is defined by $\pi(E_1) = \pi(E_2) = E'$ and $\pi(R) = R$ for any $R \neq E_1, E_2$. And a map $\phi : \mathrm{Col}_X(D') \to \mathrm{Col}_X(D)$ is defined by $\phi(c') = c' \circ \pi$.

Lemma 13. *The map ϕ is injective.*

Proof. If $\phi(c'_1) = \phi(c'_2)$, then $c'_1 \circ \pi = c'_2 \circ \pi$. Since π is a surjective map, $c'_1 = c'_2$. Therefore, ϕ is injective. □

It is easily seen that ϕ is linear when $X = R_p$. And by Lemma 13, $\mathrm{Col}_p(D') \cong \phi(\mathrm{Col}_p(D'))$. In the proof of Lemma 10, we have $\mathrm{Col}_p(D) \cong \langle h_1, \ldots, h_k \mid r_1, \ldots, r_s\rangle_p$ where $h_1, \ldots, h_k$ are the sheets of D and $r_1, \ldots, r_s$ are the relators derived from the double point curves $d_1, \ldots, d_s$ of D. Now, $E_1 = h_{j_1}$ and $E_2 = h_{j_2}$ for some $j_1, j_2 \in \{1, \ldots, k\}$. Since $\mathrm{Col}_p(D') \cong \phi(\mathrm{Col}_p(D')) \cong \langle h_1, \ldots, h_k \mid r_1, \ldots, r_s,\ h_{j_1} = h_{j_2}\rangle_p$, we have the following proposition.

Proposition 14. *Let S, S' be surface links such that S' is obtained from S by attaching a 1-handle. Let m be the integer with $\mathrm{Col}_p(S) \cong \mathbf{Z}_p^m$. Then, $\mathrm{Col}_p(S')$ is a linear subspace of $\mathrm{Col}_p(S)$ such that $\mathrm{Col}_p(S') \cong \mathbf{Z}_p^m$ or $\mathbf{Z}_p^{m-1}$.*

Example 15. Let S be a spun trefoil and D be the diagram of S illustrated in Fig. 2(1). Let D' and D'' be the diagrams illustrated in Fig. 2(2) and (3), respectively. They are diagrams of surface knots obtained from S by attaching a 1-handle. Let $h_1, \ldots, h_4, r_1, r_2, r_3$ be as in Example 11 and let r_4, r_5 be relators such that $r_4 = h_1 - h_2$, $r_5 = h_1 - h_4$. Then, $\mathrm{Col}_3(D') \cong \phi(\mathrm{Col}_3(D')) \cong \langle h_1, h_2, h_3, h_4 \mid r_1, r_2, r_3, r_4\rangle_3$ and $\mathrm{Col}_3(D'') \cong \phi(\mathrm{Col}_3(D'')) \cong \langle h_1, h_2, h_3, h_4 \mid r_1, r_2, r_3, r_5\rangle_3$. Now, we consider $(4,4)$-matrices B and C over $\mathbf{Z}_3$ given by:

$$B = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

Then, $\dim\langle r_1, r_2, r_3, r_4\rangle_3 = \operatorname{rank} B = 3$ and $\dim\langle r_1, r_2, r_3, r_5\rangle_3 = \operatorname{rank} C = 2$. Therefore, $\mathrm{Col}_3(D') \cong \mathbf{Z}_3$ and $\mathrm{Col}_3(D'') \cong \mathbf{Z}_3^2$.

Lemma 16. *Let S and S' be surface links. Let m and m' be integers such that $\mathrm{Col}_p(S) \cong \mathbf{Z}_p^m$ and $\mathrm{Col}_p(S') \cong \mathbf{Z}_p^{m'}$, respectively. Then, $\mathrm{Col}_p(S \sharp S') \cong \mathbf{Z}_p^{m+m'-1}$.*

Proof. Since $\mathrm{Col}_p(S \amalg S') \cong \mathrm{Col}_p(S) \oplus \mathrm{Col}_p(S') \cong \mathbf{Z}_p^{m+m'}$, by Proposition 14, we have $\mathrm{Col}_p(S \sharp S') \cong \mathbf{Z}_p^{m+m'-1}$ or $\mathbf{Z}_p^{m+m'}$. We consider a p-coloring C of $S \amalg S'$ such that $C(E) = 0$ if $E \in \Sigma(D)$, $C(E') = 1$ if $E' \in \Sigma(D')$ where D and D' are diagrams of S and S', respectively. By Lemma 13, $C \notin \phi(\mathrm{Col}_p(S \sharp S'))$, and hence $\mathrm{Col}_p(S \sharp S') \ncong \mathbf{Z}_p^{m+m'}$. (This implies that the relator $h_{j_1} = h_{j_2}$ in the paragraph above Proposition 14 is not a consequence of the relators derived from the double point curves of $D \amalg D'$.) Therefore, $\mathrm{Col}_p(S \sharp S') \cong \mathbf{Z}_p^{m+m'-1}$. □

Proposition 17. *Let $S_1, \ldots, S_w$ be surface links whose component numbers are $n_1, \ldots, n_w$, respectively, and let $m_1, \ldots, m_w$ be integers such that for each i, the set of p-colorings of S_i is isomorphic to $\mathbf{Z}_p^{m_i}$. Then, $(m_1 + \cdots + m_w) - (n_1 + \cdots + n_w) \leqslant u(S_1 \sharp \cdots \sharp S_w)$.*

Proof. Put $m = m_1 + \cdots + m_w$ and $n = n_1 + \cdots + n_w$. Let S be the connected sum $S_1 \sharp \cdots \sharp S_w$. If $m - n - 1 \geqslant u(S)$, then there is a set of $m - n - 1$ 1-handles such that the surface link S' obtained from S by attaching these 1-handles is an unknotted surface link. Since the component number of S is $n - (w - 1)$, the component number of S' is at most $n - (w - 1)$. Hence, $|\mathrm{Col}_p(S')| \leqslant p^{n-(w-1)}$. On the other hand, by Lemma 16, $|\mathrm{Col}_p(S)| = p^{m-(w-1)}$. By Proposition 14, $|\mathrm{Col}_p(S')| \geqslant p^{(m-(w-1))-(m-n-1)} = p^{n-(w-2)}$. This is a contradiction. □

Remark 18. For a surface link S, $\mathrm{Col}_p(S)$ is related to a homology group of a double branched cover for S. Specifically, in [25], the relation between core group and fundamental group of double branched cover for S is discussed, and by abelianizing, it gives a relation between $\mathrm{Col}_p(S)$ and the homology of double branched cover for S. (Such a relation is given by Fox in the classical case.) Then, Miyazaki's [22] inequality $\rho(S) \leqslant u(S)$ will related to the exponent m_i in Proposition 17, since the case $t = -1$ that Miyazaki uses at the bottom of p. 83 in [22] would be related to the rank of homology of the double cover. Thus, Proposition 17 will also follow from Miyazaki's inequality.

Corollary 19. *For each i with $1 \leqslant i \leqslant w$, let S_i be a surface knot with $u(S_i) = 1$ such that $|\mathrm{Col}_p(S_i)| = p^2$. Then, $u(S_1 \sharp \cdots \sharp S_w) = w$.*

Proof. By Proposition 17, $2w - w = w \leqslant u(S_1 \sharp \cdots \sharp S_w)$. On the other hand, $u(S_1 \sharp \cdots \sharp S_w) \leqslant u(S_1) + \cdots + u(S_w) = w$. Thus, $u(S_1 \sharp \cdots \sharp S_w) = w$. □

3. Quandle cocycle invariants

Let X be a quandle and fix the associated group. In [6], the quandle homology was defined to construct invariants of classical knots or surface links. N. Andruskiewitsch and M. Graña [1] provided generalizations of quandle homology theory. Now, we review quandle homology theory of G_X-module (cf. [1,5,7,28]). The original idea of this theory appeared in [9] and in Section 4 of [10].

Consider the free $\mathbf{Z}G_X$-module $C_n(X) = G_X X^n$ with basis X^n for $n > 0$. Put $C_0(X) = \mathbf{Z}G_X$ and $C_n(X) = 0$ for $n < 0$. We define $\partial_n : C_n(X) \to C_{n-1}$ by

$$\partial_n(x_1, \ldots, x_n) = (-1)^n \sum_{i=1}^{n} \big[(-1)^i [x_i, x_{i+1}, \ldots, x_n](x_1, \ldots, \hat{x}_i, \ldots, x_n) \\ - (-1)^i (x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n)\big]$$

for $n > 1$, $\partial_1(x_1) = -x_1 + 1$, and $\partial_n = 0$ for $n < 1$, where

$$[x_1, x_2, \ldots, x_n] = \big(\big(\cdots(x_1 * x_2) * x_3\big) * \cdots\big) * x_n.$$

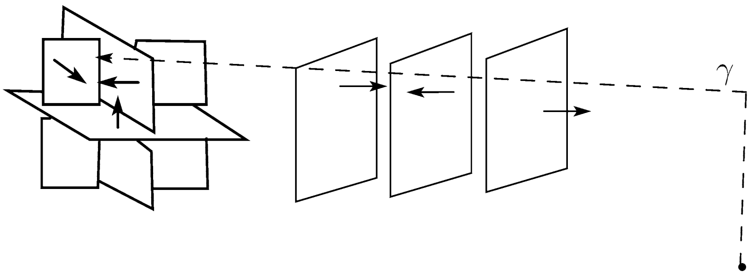

Fig. 3.

In particular, the 3-cocycle condition for a 3-cochain κ is written as

$$\begin{aligned} &w\kappa_{x,y,z} + \kappa_{x*z,y*z,w} + \big((y*z)*w\big)\kappa_{x,z,w} + \kappa_{y,z,w} \\ &\quad = \big(\big((x*y)*z\big)*w\big)\kappa_{y,z,w} + \kappa_{x*y,z,w} + (z*w)\kappa_{x,y,w} + \kappa_{x*w,y*w,z*w} \end{aligned}$$

for any $x, y, z, w \in X$ where $\kappa_{x,y,z} = \kappa(x, y, z)$. We call this condition a *rack* 3*-cocycle condition*. When κ further satisfies $\kappa_{x,x,y} = \kappa_{x,y,y} = 0$, we call κ a *quandle* 3*-cocycle*, or a 3*-cocycle*.

Let D be a diagram of a surface link S. Let γ be an arc from the region at infinity of $\mathbf{R}^3$ to the target region of a triple point t. Assume that γ intersects D transversely in some points thereby missing double point curves, branch points and triple points (Fig. 3). Let h_i, $i = 1, \dots, k$, in this order, be the sheets of D that intersect γ from the region at infinity to the triple point t. Let X be a finite quandle and let κ be a 3-cocycle valued in a G_X-module M. For a coloring C, we define the *Boltzmann weight* at the triple point t by

$$B(C,t) = \varepsilon(t)\big(C(h_1)^{\varepsilon(h_1)}C(h_2)^{\varepsilon(h_2)}\cdots C(h_k)^{\varepsilon(h_k)}\big)\kappa_{x,y,z} \in M,$$

where x, y and z are the sheets around t such that z is the top sheet, y is the middle sheet from which the orientation normal of z points, and x is the bottom sheet from which the orientation normals of y and z point. The sign $\varepsilon(t)$ is the sign of the triple point t. The exponent $\varepsilon(h_i)$ is 1 if the arc γ crosses the sheet h_i against its normal, and is -1 otherwise, for $i = 1, \dots, k$. The value $B(C,t)$ does not depend on the choice of γ. The family $\Phi_\kappa(S) = \{\sum_t B(C,t)\}_{C \in \mathrm{Col}_X(D)}$ is called the *quandle cocycle invariant* with respect to 3-cocycle κ (cf. [5,7]), where $\sum_t$ is taken over all crossing of D. It does not depend on the choice of diagram D of the surface link S.

For two multi-sets A' and A'', we use notation $A' \overset{m}{\leqslant} A''$ when $g \in A'$ implies $a_g(A') \leqslant a_g(A'')$, where $a_g(A)$ is the number of g in A. (This is different from the one in [5,7].)

Let S be a surface link and D be a diagram of S. The *triple point number* of D, $t(D)$, means the number of triple points of D. The *triple point number* of S, $t(S)$, is the minimal number of $t(D)$ among all diagrams D of S.

Lemma 20. *Let S be a surface link and κ be a* 3*-cocycle. If there is a non-zero element of M in $\Phi_\kappa(S)$, then $t(S) \geqslant 1$, and hence $\tau(S) \geqslant 1$.*

Proof. If $t(S) = 0$, then all elements of $\Phi_\kappa(S)$ are 0. □

Lemma 21. *Let S and S' be surface links such that S' is obtained from S by attaching a finite number of* 1*-handles. Then, $\Phi_\kappa(S') \overset{m}{\leqslant} \Phi_\kappa(S)$.*

Proof. Let D, D' and H be as in Lemma 12. We may identify the triple points $\{t_1, \dots, t_u\}$ of D with that of D'. Let γ_i be arcs in $\mathbf{R}^3$ from the region at infinity to the triple point t_i for any i with $1 \leqslant i \leqslant u$ such that γ_i intersects D transversely in some points thereby missing double point curves, branch points, triple points and 1-handle H. Let ϕ be the map as in Section 2. Then $\sum_{t_i} B(c', t_i) = \sum_{t_i} B(\phi(c'), t_i)$. Since ϕ is injective (Lemma 13), $\Phi_\kappa(S') \overset{m}{\leqslant} \Phi_\kappa(S)$. □

4. Proofs of main results and examples

Proof of Theorem 2. If $l \geqslant \tau(S)$, there is a set of l 1-handles such that the surface link S' obtained from S by attaching these 1-handles is a pseudo-ribbon surface link, i.e. $t(S') = 0$. By Proposition 14, $|\operatorname{Col}_p(S')| \geqslant p^{m-l}$. On the other hand, $\Phi_\kappa(S') \overset{m}{\leqslant} \Phi_\kappa(S)$ by Lemma 21, and hence $a_0(\Phi_\kappa(S)) \geqslant a_0(\Phi_\kappa(S'))$. By assumption, $|\operatorname{Col}_p(S')| \geqslant p^{m-l} > a_0(\Phi_\kappa(S)) \geqslant a_0(\Phi_\kappa(S'))$. Therefore, there are colorings which contribute non-zero elements of M in $\Phi_\kappa(S')$. By Lemma 20, $t(S') \geqslant 1$. This is a contradiction. □

Proof of Theorem 3. By assumption, $O_\kappa(S \amalg S')$ is a subspace of $\operatorname{Col}_p(S \amalg S')$, and hence $a_0(\Phi_\kappa(S \amalg S')) = p^s$ for some integer s. Applying the same argument as in Section 2 to $O_\kappa(S \amalg S')$, we have $a_0(\Phi_\kappa(S \sharp S'))(= |O_\kappa(S \sharp S')|) = p^s$ or p^{s-1}. Consider a p-coloring C of $S \amalg S'$ such that $C(E) = 0$ if $E \in \Sigma(D)$, $C(E') = 1$ if $E' \in \Sigma(D')$ where D and D' are diagrams of S and S', respectively. Then C contributes $0 \in M$ in $\Phi_\kappa(S \amalg S')$. Therefore, $|\operatorname{Col}_p(S \sharp S')| = p^{m+m'-1}$ and $a_0(\Phi_\kappa(S \sharp S')) = p^{s-1}$. If $p^{m+m'-l} > a_0(\Phi_\kappa(S \amalg S'))(= p^s)$, then $p^{(m+m'-1)-l} > p^{s-1}(= a_0(\Phi_\kappa(S \sharp S'))$. By Theorem 2, the inequality $l + 1 \leqslant \tau(S \sharp S')$ holds. □

Proofs of Corollary 6 and 7. Let D and D' be diagrams of S and S', respectively. By combination of trivial colorings of D and D', the number of p-colorings that contribute 0 in $\Phi_{\theta_p}(S \amalg S')$ is at least p^2. By assumption, $O_\kappa(S \amalg S') \cong \mathbf{Z}_p^2$. Applying Theorem 3 to S and S' with $m = m' = 2$ and $l = 1$, we have $2 \leqslant \tau(S \sharp S')$. On the other hand, $\tau(S \sharp S') \leqslant u(S \sharp S') \leqslant u(S) + u(S') = 2$. We have Corollary 6. By a similarly argument, we have Corollary 7. □

For twist spun 2-bridge knots, the following proposition have been known. (See [29] for the definition of twist spun knots.)

Proposition 22. [18,20,27] *Let S be an r-twist spun 2-bridge knot. Then,*

(1) $\tau(S) = 0$ *and* $u(S) = 1$ *for* $r = 0$.
(2) $\tau(S) = u(S) = 1$ *for* $r \geqslant 2$.

Example 23. Let σ_α be a spun 2-bridge knot $S(\alpha, 1)$ in Schubert form for odd α. If $(\alpha, \alpha') \neq 1$, then there is an odd prime q that is a divisor common to α and α'. It is easy to see that $|\operatorname{Col}_q(\sigma_\alpha)| = |\operatorname{Col}_q(\sigma_{\alpha'})| = q^2$. On the other hand, by Proposition 22(1), $u(\sigma_\alpha) = u(\sigma_{\alpha'}) = 1$. By Corollary 19, $u(\sigma_\alpha \sharp \sigma_{\alpha'}) = 2$. Furthermore, this example also appeared in [22, p. 83, Remark 2] .

Example 24. (1) *Mochizuki's* 3*-cocycle* θ_p valued in $\mathbf{Z}_p$, which is a generator of the third quandle cohomology group of R_p with trivial action, was given in [3,23]. By [6,26], we have $|\operatorname{Col}_3(T_r)| = |\operatorname{Col}_3(T_{r+6})| = a_0(\Phi_{\theta_3}(T_r \amalg T_{r+6})) = 3^2$ where T_r is the r-twist spun trefoil for an even integer r with $r \not\equiv 0 \pmod 6$. On the other hand, by Proposition 22(2), $u(T_r) = u(T_{r+6}) = 1$. By Corollary 6, we have $\tau(T_r \sharp T_{r+6}) = u(T_r \sharp T_{r+6}) = 2$.

(2) Associated with θ_p, the quandle cocycle invariants of twist spun 2-bridge knots are calculated in [14]. By an argument similarly to (1), we have triple point cancelling numbers of some of 2-knots that are connected sum of twist spun 2-bridge knots. For example, we have $\tau(F_r \sharp F_{r'}) = u(F_r \sharp F_{r'}) = 2$ for even numbers r and r' with $r \equiv 2, 8 \pmod{10}$ and $r' \equiv 4, 6 \pmod 6$ where F_r is the r-twist spun figure eight knot.

Example 25. In [5,7], a 3-cocycle κ of R_3 with wreath product action was given, where κ is valued in $\mathbf{Z}^3$. And the cocycle invariants of twist spun 3-colorable knots (up to 9-crossing) associated with κ was calculated. According to their calculation, we have $|\operatorname{Col}_3(S)| = 3^2, a_0(\Phi_\kappa(\coprod_n S)) = 3^n$ or $|\operatorname{Col}_3(-S)| = 3^2, a_0(\Phi_\kappa(\coprod_n(-S))) = 3^n$ for a $2r$-twist spin of S of $7_7, 9_{11}, 9_{15}$ or 9_{17} with $r \neq 0$. (These classical knots are 2-bridge knots.) On the other hand, by Proposition 22(2), $u(S) = 1$. Since $\tau(K) = \tau(-K)$ and $u(K) = u(-K)$ for any surface link K, by Corollary 7, we have $\tau(\sharp_n S) = u(\sharp_n S) = n$.

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